Transverse Vibration of a Uniform Euler-Bernoulli Beam Under Varying Axial Force Using Differential Transformation Method

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This paper presents the application of techniques of differential transformation method (DTM) to analyze the transverse vibration of a uniform Euler-Bernoulli beam under varying axial force. The governing differential equation of the transverse vibration of a uniform Euler-Bernoulli beam under varying axial force is derived and verified. The varying axial force was extended to the more general case which was high polynomial consisted of many terms. The concepts of DTM were briefly introduced. Numerical calculations are carried out and compared with previous published results. The accuracy and the convergence in solving the problem by DTM are discussed.

Key Words: Euler-Bernoulli Beam, Differential Transformation, Varying Axial Force

1. Introduction

The structural elements of a mechanism are often subjected to axially distributed force. Among those structural elements are a tie-bar under a constant axial force, a vertically oriented uniform beam in a gravity field subjected to a linearly distributed axial force, and a beam in a centrifugal field subjected to parabolic distribution. Especially, there are many cases of beams with arbitrary cross-section subjected to nonlinearly distributed force.

McCallion and Bokian investigated the transverse vibration of uniform tie-bars (McCallion, 1973; Bokaian, 1988; 1990). Schafer investigated the transverse vibration of 'hanging' and 'standing' uniform cantilevers taking account of the self-weight by the Rayleigh-Ritz method (Schafer, 1985). Fauconneau and Laird obtained upper bound eigen-frequencies of a simply supported uniform beam under a linearly varying compressive or tensile axial force also by using the Rayleigh-Ritz method (Fauconneau and Laird, 1976). Yokoyama studied the vibration of 'hanging' Timoshenko beam under gravity by using the finite element method (Yokayama, 1990). Naguleswaran obtained the first three dimensionless natural frequencies of a uniform cantilever with the clamped-free and the free-clamped boundary conditions by using the Frobenius method ('hanging' and 'standing') (Naguleswaran, 1991). The general solution consisted of the superposition of four linearly independent power series solution functions. Naguleswaran (2004) studied the transverse vibration and investigated the dynamic instability of a uniform Euler-Bernoulli beam under linearly varying axial force for 16 combinations of classical boundary conditions by the Frobenius method. The axial tension distribution consists of a constant part and a part proportional to axial co-ordinate. These are two system parameters. Wholly tensile, partly tensile and wholly compressive axial force distribution are considered. Chung and Park investigated dynamic characteristics of a beam subjected to an axial force and a force of time dependent frequency (Chung and Park, 1986).

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The transformation technique called Differential Transformation Method (DTM) is applied to the analysis of the transverse vibration of a uniform Euler-Bernoulli beam under varying axial force (Zhou, 1986). The varying axial force was extended to the more general case which was presented in high polynomial. The concept of this transformation was applied to solve linear and nonlinear initial value problems in electric circuit analysis. Recently this method seems to attract researcher's interest in solving eigenvalue problems (Malik and Dang, 1998; Kuang and Ho, 1996). In this study, the concepts of DTM were briefly introduced. DTM was applied to the analysis of the transverse vibration of a uniform Euler-Bernoulli beam under varying axial force. Numerical calculations are carried out and compared with previous published results. These results can be used as the base data for the dynamic design of beam. The accuracy and the convergence in solving the problem by DTM are discussed.

2. Differential Transformation Method

The concept of this transformation was first proposed by Zhou (1986). Differential Transformation Method (DTM) is based on Taylor series expansion and the solution of differential equations is obtained through recursive algebraic equation of the transformed governing equation of motion by its basic mathematical operations. DTM is a very useful method for solving linear and non-linear differential problems.

Let $y(x)$ be analytic in domain D and $x=x_0$ be a point in D. Then there exists precisely one power series with center at $x=x_0$ which represents $y(x)$; this series, the Taylor series of the function $y(x)$, is as following form

$$y(x)=\sum_{k=1}^{\infty}\frac{(x-x_0)^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x^0} \text{ for } \forall x \subset D \quad (1)$$

If we define differential transformation of function $Y(k)$ as follows

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x^0} \quad (2)$$

Table 1 Basic operations of the differential transformation

Original function	T-function
$w(x)=y(x)\pm z(x)$	$W(k)=Y(k)\pm Z(k)$
$z(x)=\lambda y(x)$	$Z(k)=\lambda Y(k)$
$w(x)=\frac{d^n y(x)}{dx^n}$	$W(k)=(k+1)(k+2)\cdots(k+n)Y(k+n)$
$w(x)=y(x)z(x)$	$W(k)=\sum_{l=0}^{k}Y(l)Z(k-l)$
$w(x)=x^m$	At $k\leq m$ $W(k)=\frac{{}_m\mathrm{P}_k}{k!}x_0^{m-k}$ At $k>m$ $W(k)=0$
$w(x)=\sin(\lambda x)$	$W(k)=\frac{\lambda^k}{k!}\sin\left(\frac{\pi k}{2}\right)$ at $x_0=0$

and substituting Eq. (2) into Eq. (1) and rearranging, the original function $y(x)$ can be obtain as

$$y(x)=\sum_{k=1}^{\infty}(x-x_0)^k Y(k) \quad (3)$$

where $Y(k)$ is called T-function for the original function $y(x)$, and Eq. (3) is the differential inverse transformation of $Y(k)$.

From the above definition of the differential transformation, we can derive the rules of transformational operations: some examples of these, which are useful in the following analysis, are presented in Table 1.

Especially, if the original function is $w(x)=x^m$ when $x_0=0$, it's T-function is represented as for $x_0=0$

$$W(k)=\delta(k-m)=\begin{cases}1, & k=m\\ 0, & k\neq m\end{cases} \quad (4)$$

For actual applications, $y(x)$ is approximately represented by Eq. (5) which takes finite terms.

$$y(x)=\sum_{k=1}^{n}(x-x_0)^k Y(k) \quad (5)$$

where n is natural number decided by convergence of solution.

3. Governing Equations and Boundary Conditions

Figure 1 shows the Euler-Bernoulli beam under varying axial force. In the Fig. 1, EI, m and

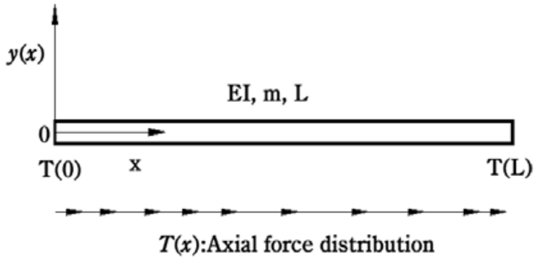

Fig. 1 The beam coordinate system and axial force distribution

L represent the flexural rigidity, mass per unit length and length of the uniform beam respectively. The end $(x=0)$ is axially restrained while the opposite end is axially free. The varying axial force distribution $\bar{T}(x)$ at abscissa x along the axis of the beam is

$$\bar{T}(x)=\bar{T}(0)+\sum_{n=1}^{p}\alpha_n\left(\frac{x}{L}\right)^n \tag{6}$$

where $\bar{T}(0)$ is the axial force at $x=0$. And, α_n is the coefficient of axial force distribution function, and p is integer.

Positive axial force and negative axial force are tensile and compressive respectively.

If at abscissa x, $y(x)$, ω, $M(x)$ and $Q(x)$ are transverse deflection of the beam, the free vibration frequency, the amplitudes of bending moment and shearing force respectively. The equilibrium conditions of a differential element are represented as following

$$M(x)=EI\frac{d^2y(x)}{dx^2} \tag{7}$$

$$Q(x)+\frac{dM(x)}{dx}-\bar{T}(x)\frac{dy(x)}{dx}=0 \tag{8}$$

$$\frac{dQ(x)}{dx}+m\omega^2y(x)=0 \tag{9}$$

Substituting Eq. (7) into Eq. (8), substituting Eq. (8) into Eq. (9) and rearranging, the governing equation of the Euler-Bernoulli beam under varying axial force can be obtain as following

$$\begin{aligned} &EI\frac{d^4y(x)}{dx^4}-\frac{d\bar{T}(x)}{dx}\frac{dy(x)}{dx}\\ &-\bar{T}(x)\frac{d^2y(x)}{dx^2}-m\omega^2y(x)=0 \end{aligned} \tag{10}$$

We introduce the dimensionless variables X and $y(X)$, the dimensionless varying axial force distribution $T(X)$, the frequency parameter β and the variable axial parameters γ_n as follows

$$\begin{aligned} &X=\frac{x}{L},\ \beta^4=\frac{m}{EI}\omega^2L^2,\\ &T(X)=\tau_0+\sum_{n=1}^{p}\gamma_nX,\ T_1(X)=\frac{dT(X)}{dX} \end{aligned} \tag{11}$$

where $\tau_0=\frac{\bar{T}(0)L^2}{EI}$ (the constant axial force parameter at $x=0$) and $\gamma_n=\frac{\alpha_nL^2}{EI}$.

Substituting Eq. (11) into Eq. (10), the following governing equation can be obtained

$$\begin{aligned} &\frac{d^4y(X)}{dX^4}-T_1(X)\frac{dy(X)}{dX}\\ &-T(X)\frac{d^2y(X)}{dX^2}-\beta^4y(X)=0 \end{aligned} \tag{12}$$

In order to solve the Eq. (12), four boundary conditions are needed. These can be obtained by specifying two boundary conditions at one end $X=0$ and two boundary conditions at the other end $X=1$. The boundary conditions considered here are as follows:

Clamped end (Cl):

$$y(X)=0 \text{ and } \frac{dy(X)}{dX}=0 \tag{13}$$

Pinned end (Pn):

$$y(X)=0 \text{ and } M(X)=0 \tag{14}$$

Sliding end (Sl):

$$\frac{dy(X)}{dX} \text{ and } Q(X)=0 \tag{15}$$

Free end (Fr):

$$M(X) \text{ and } Q(X)=0 \tag{16}$$

4. Application of Differential Transformation

Taking differential transformation of Eq. (12) and using Table 1 mentioned above, we can obtain

$$(k+1)(k+2)(k+3)(k+4)\bar{Y}(k+4) - \sum_{l=1}^{k}\bar{T}_1(l)(k-l+1)\bar{Y}(k-l+1) - \sum_{l=1}^{k}\bar{\bar{T}}(l)(k-l+1)(k-l+2)\bar{Y}(k-l+2) - \beta^4\bar{Y}(k)=0 \quad (17)$$

where $\bar{Y}$, $\bar{T}_1$ and $\bar{\bar{T}}$ are T-functions of $y(X)$, $T_1(X)$ and $T(X)$ respectively.

In order to analyze the Euler-Bernoulli beam under varying axial force, the boundary conditions; from Eq. (13) to Eq. (16); should be transformed. The transformed boundary condition equations at each end should be obtained by differential transformation method as follows

At the end $X=0$

Clamped end (Cl):

$$\bar{Y}(0)=0 \text{ and } \bar{Y}(1)=0 \quad (18)$$

Pinned end (Pn):

$$\bar{Y}(0)=0 \text{ and } \bar{Y}(2)=0 \quad (19)$$

Sliding end (Sl):

$$\bar{Y}(1)=0 \text{ and } -3!\times\bar{Y}(3)+T(0)\times\bar{Y}(1)=0 \quad (20)$$

Free end (Fr):

$$\bar{Y}(2)=0 \text{ and } -3!\times\bar{Y}(3)+T(0)\times\bar{Y}(1)=0 \quad (21)$$

At the end $X=1$

Clamped end (Cl):

$$\sum_{k=0}^{n}\bar{Y}(k)=0 \text{ and } \sum_{k=0}^{n}k\bar{Y}(k)=0 \quad (22)$$

Pinned end (Pn):

$$\sum_{k=0}^{n}\bar{Y}(k)=0 \text{ and } \sum_{k=0}^{n}k(k-1)\bar{Y}(k)=0 \quad (23)$$

Sliding end (Sl):

$$\sum_{k=0}^{n}k\bar{Y}(k)=0 \text{ and } \sum_{k=0}^{n}k(k-1)(k-3)\bar{Y}(k)+T(1)\times\sum_{k=0}^{n}k\bar{Y}(k)=0 \quad (24)$$

Free end (Fr):

$$\sum_{k=0}^{n}k(k-1)\bar{Y}(k)=0 \text{ and } \sum_{k=0}^{n}k(k-1)(k-3)\bar{Y}(k)+T(1)\times\sum_{k=0}^{n}k\bar{Y}(k)=0 \quad (25)$$

5. Numerical Analysis and Discussions

In order to analyze the Euler-Bernoulli beam under varying axial force, we should take advantage of the transformed Eq. (17) and the four corresponding boundary condition Eqs. (18) ~ (25). These equations can be represented in following matrix form

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} & a_{1,k+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} & a_{2,k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k} & a_{k+1,k+1} \end{bmatrix} \begin{bmatrix} \bar{Y}(0) \\ \bar{Y}(1) \\ \vdots \\ \bar{Y}(k) \end{bmatrix} = 0 \quad (26)$$

A non-trivial solution exists when the determinant of the coefficient matrix vanishes. This condition leads to the following frequency equation:

$$\det \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} & a_{1,k+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} & a_{2,k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k} & a_{k+1,k+1} \end{vmatrix} = 0 \quad (27)$$

where $a_{i,j}$ are functions of the variables β, τ_0 and γ_n.

From the above equation, the natural frequency parameter equation and the buckling parameter equation of the Euler-Bernoulli beam under varying axial force can be obtained.

Numerical analysis of the Euler-Bernoulli beam under varying axial force is performed for sixteen boundary conditions, which were chosen from the Eqs. (13) ~ (16), and $m=2$ ($T(X)=\tau_0+\gamma_1$). The natural frequencies are obtained in five significant figures and compared to those of the reference (Naguleswaran, 2004).

In the Table 2 the natural frequencies represented by the first three frequency parameters (β) are given for the case $\tau_0=10.0$ and γ_1 stated

Table 2 Comparison of the first three frequency parameters for τ_0=10.0 and various γ_1

BC		γ_1=100.0			γ_1=4.0			γ_1=−3.0		
		β_1	β_2	β_3	β_1	β_2	β_3	β_1	β_2	β_3
Cl/Cl	Frobenius	5.8768	8.9720	11.9595	5.0437	8.1234	11.2122	4.9587	8.0475	11.1503
	DTM	5.8768	8.972	11.9595	5.0437	8.1234	11.2122	4.9587	8.0475	11.1503
Cl/Pn	Frobenius	5.5709	8.5122	11.3923	4.4144	7.4147	10.4694	4.2694	7.3077	10.3898
	DTM	5.5709	8.5122	11.3932	4.4144	7.4147	10.4694	4.2647	7.3077	10.3898
Cl/Sl	Frobenius	3.5912	7.0372	9.9341	2.8569	5.9054	8.9358	2.7525	5.7828	8.8455
	DTM	3.5912	7.0372	9.9341	2.8569	5.9054	8.9358	2.7525	5.7828	8.8455
Cl/Fr	Frobenius	3.5876	6.9736	9.7435	2.7660	5.4542	8.3161	2.5956	5.2063	8.1548
	DTM	3.5876	6.9736	9.7435	2.7660	5.4542	8.3161	2.5956	5.2063	8.1548
Pn/Cl	Frobenius	5.2505	8.2814	11.2405	4.3835	7.4000	10.4614	4.2906	7.3192	10.3960
	DTM	5.2505	8.2814	11.2405	4.3835	7.4000	10.4614	4.2906	7.3192	10.396
Pn/Pn	Frobenius	5.0032	7.8617	10.6990	3.8322	6.7141	9.7281	3.6689	6.5970	9.6426
	DTM	5.0032	7.8617	10.6990	3.8322	6.7141	9.7281	3.6689	6.5970	9.6426
Pn/Sl	Frobenius	3.0737	6.4101	9.2566	2.4084	5.2461	8.2099	2.3111	5.1127	8.1122
	DTM	3.0737	6.4101	9.2566	2.4084	5.2461	8.2099	2.3111	5.1127	8.1122
Pn/Fr	Frobenius	3.0722	6.3678	9.1013	2.3588	4.8738	7.6238	2.2170	4.6126	7.4454
	DTM	3.0722	6.3678	9.1013	2.3588	4.8738	7.6238	2.217	4.6126	7.4454
Sl/Cl	Frobenius	3.8567	6.837	9.7732	2.8713	5.8892	8.9265	2.7415	5.7959	8.8526
	DTM	3.8567	6.837	9.7732	2.8713	5.8892	8.9265	2.7415	5.7959	8.8526
Sl/Pn	Frobenius	3.7150	6.5037	9.2877	2.4770	5.2530	8.2117	2.2478	5.1071	8.1108
	DTM	3.7150	6.5037	9.2877	2.4770	5.253	8.2117	2.2478	5.1071	8.1108
Sl/Sl	Frobenius	5.0483	7.8632	10.6986	3.8325	6.7141	9.7281	3.6691	6.5970	9.6426
	DTM	5.0483	7.8632	10.6986	3.8325	6.7141	9.7281	3.6691	6.5970	9.6426
Sl/Fr	Frobenius	5.0346	7.7720	10.4718	3.6296	6.2133	9.0873	3.3514	5.9897	8.9404
	DTM	5.0346	7.7720	10.4718	3.6296	6.2133	9.0873	3.3514	5.9897	8.9404
Fr/Cl	Frobenius	3.7806	6.3957	9.1596	2.7547	5.3778	8.2704	2.6142	5.2739	8.1912
	DTM	3.7806	6.3957	9.1596	2.7547	5.3778	8.2704	2.6142	5.2739	8.1912
Fr/Pn	Frobenius	3.6551	6.1204	8.7159	2.4090	4.8209	7.5858	2.1752	4.6632	7.4765
	DTM	3.6551	6.1204	8.7159	2.409	4.8209	7.5858	2.1752	4.6632	7.4765
Fr/Sl	Frobenius	4.8344	7.3614	10.0767	3.5792	6.1613	9.0567	3.4052	6.0345	8.9647
	DTM	4.8344	7.3614	10.0767	3.5792	6.1613	9.0567	3.4052	6.0345	8.9647
Fr/Fr	Frobenius	4.8247	7.2942	9.8855	3.4222	5.7311	8.4498	3.1479	5.4969	8.2900
	DTM	4.8247	7.2942	9.8855	3.4222	5.7311	8.4498	3.1479	5.4969	8.2900

in first row. And, Cl, Pn, Sl and Fr designated as Clamped, Pinned, Sliding and Free boundary conditions respectively. The results obtained by using DTM agree with those results obtained by using Frobenius method (Naguleswaran, 2004).

6. Conclusions

In this paper, the Differential Transformation Method is applied to the analysis of the transverse vibration of a uniform Euler-Bernoulli beam under varying axial force. The varying axial force was extended to the more general case which was represented in high polynomial consisted of many terms. The calculated natural frequency parameters are compared to the reference and the results are as follows:

(1) The results obtained by DTM agree with those obtained by Frobenius method (Naguleswaran, 2004),

(2) Differential Transformation Method can be used as an alternative method to solve the differential equation problems in addition to Finite Element Method (FEM), Finite Differential Method (FDM) and Frobenius method.

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